is uncommon. The usual German expression is "vollständige Induktion." In criticism of this term Federigo Enriques[1] says: "We should not confound *mathematical induction,* namely the argument from n to $n + 1 \cdots$ with the *complete induction* of Aristotle." In this Aristotelian sense the term "vollständige Induktion" is used in 1840 in the article "Induction" in Ersch and Gruber's *Encyklopädie,* where we find the example: If two sides are found to be greater than the third side in plane triangles with a right angle, and with an obtuse angle, and also with only acute angles, and this inequality is shown to be true likewise of spherical triangles, then the inequality can be asserted to be true of *all* triangles. Here a "vollständige Induktion" is quite different from the argument from n to $n + 1$. The use of the same name for two different types of induction is objectionable. The name "vollständige Induktion" was used by R. Dedekind in his *Was sind und was sollen die Zahlen,* 1887, §§ 59 and 80. Through him the method received great emphasis in Germany.[2] The English equivalent of "vollständige Induktion," namely, "complete induction," is seldom used.[3] According to A. Haas[4] the designation "höhere Induktion" is also employed. Poincaré, in his *Science et Hypothèse,* does not restrict himself to any one name, but is partial to the phrases "démonstration par récurrence," "raisonnement par récurrence."

MISCELLANEA.

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§ I. Concerning Greatest and Least Absolute Values of Analytic Functions.

The following theorems are known:

I*a*. If $f(z)$ is an analytic (single-valued) function of the complex variable $z = x + iy$, regular in the interior and on the boundary of a circle C about a point a of the complex plane as center, then $|f(a)| \leqq M$, where M is the largest value which $|f(z)|$ assumes on the boundary of C.

I*b*. Under the same conditions for $f(z)$, $|f(c)| \leqq M$, where c is any interior point of the circle C.[5]

I*c*. In I*b* the circular region may be replaced by a region S, not necessarily simply connected, but the boundary of which we shall assume, in this as in the

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[1] F. Enriques, *Principles of Science,* transl. by K. Royce, Chicago and London, 1914, p. 133.

[2] An excellent article in the English language on "Mathematical Induction" is that by C. J. Keyser in the *Americana.*

[3] See, however, W. H. Bussey, "The Origin of Mathematical Induction," Am. Math. Monthly, Vol. 24, 1917, p. 199.

[4] A. Haas, *Lehrbuch über den binomischen und polynomischen Lehrsatz,* Bremerhaven, 1906, p. 6.

[5] For I*a* and I*b* compare, among others: Fouët, *Leçons sur la théorie des fonctions analytiques,* 1910, Vol. II, p. 79; Osgood, *Lehrbuch der Funktionentheorie,* 1912, Vol. I, p. 300; Vivanti-Gutzmer, *Theorie der eindeutigen analytischen Funktionen,* 1906, p. 81.

following theorems, to consist of a finite number of "regular" arcs of curves.[1] For any interior point c we shall again have $|f(c)| \leqq M$, where M is the largest value which $|f(z)|$ assumes on the boundary of S.

In Ia, Ib, Ic, the sign of equality holds only when $f(z) \equiv$ const.

Of the three theorems, the last two follow immediately from the first, because it would otherwise be possible to select an interior point of the region where $|f(z)|$ assumes its largest value and to consider a small circle about this point as center, thus leading to a contradiction of Ia.

Corollary. If the absolute value of a single-valued analytic function $f(z)$ is regular along the whole boundary of a region S, and if it is possible to find in the interior of S any point for which $|f(z)|$ exceeds the largest value of $|f(z)|$ along the boundary, $f(z)$ must have at least one singular point in the interior.

It is well known that Ia, Ib, Ic also hold when in the theorems $|f(z)|$ is replaced by the real part u or by the imaginary part v of

$$f(z) = f(x + iy) = u(x, y) + i \cdot v(x, y).^2$$

As a direct consequence of Ia, we also mention the following theorem, which is of great importance in many modern investigations in function theory:

Let $C(r_1)$, $C(r_2)$, $\cdots$, $C(r_n)$ denote concentric circles about the point $z = 0$ of radii $r_1 < r_2 < \cdots < r_n$, respectively, and such that the single-valued analytic function $f(z)$ is regular in the interior and on the boundary of $C(r_n)$; let $M(r_\nu)$ denote the largest value which $|f(z)|$ assumes on the boundary of $C(r_\nu)$; then $M(r_1) \leqq M(r_2) \leqq \cdots \leqq M(r_n)$ (no signs of equality to be admitted unless $f(z) \equiv$ const.), and by a limiting process it is shown that $M(r)$, considered as a function of the continuously increasing radius r, is a monotone increasing function.[3]

Some further consequences of Ia are developed in this note. They are such obvious extensions that they must have occurred to many mathematicians, but (with the exception of II) I do not recall seeing them in print. However, no thorough study of the literature was attempted.

II. Assume $f(z)$ single-valued and regular in the interior of a region S and on the boundary, and let $m > 0$ be the smallest value which $|f(z)|$ assumes on the boundary. The existence of a point c in the interior of S such that $|f(c)| < m$ constitutes a necessary and sufficient condition that $f(z)$ have at least one zero in the interior of S.[4]

Proof: The condition is of course *necessary*. We still have to show that in case $f(z)$ has no zero in S, $|f(z)|$ assumes for some point of the boundary a smaller value than for any interior point. This is proved by applying Ic to the

[1] For the definition of "regular arc"—"reguläres Bogenstück"—see Osgood, *loc. cit.*, p. 51.

[2] See for Ic Osgood, *loc. cit.*, p. 622; Burkhardt, *Einführung in die Theorie der analytischen Funktionen* . . ., 1908, p. 125, and many others.

[3] Vivanti-Gutzmer, *loc. cit.*, p. 81; Fouët, *loc. cit.*, p. 78; Borel, *Leçons sur les fonctions entières*, 1900, p. 107; Blumenthal, *Principes de la théorie des fonctions . . . d'ordre infini*, 1910, p. 5; and others.

[4] Fouët, *loc. cit.*, p. 79 (stated for a circular region).

function $\varphi(z) = 1/f(z)$; since $f(z)$ has no zero in S or on the boundary, $\varphi(z)$ is regular in S and on the boundary, and $|\varphi(z)|$ assumes its *largest*, $|f(z)|$ its *smallest*, value, for a point on the boundary. The analogon of the corollary given above is obvious.

So far, the terms "maximum of $|f(z)|$" and "minimum of $|f(z)|$" have been avoided, because $|f(z)|$ may very well have several maxima and minima as z moves along the boundary of S. We were concerned only with the "largest" value of $|f(z)|$, the "maximum maximorum," and the "smallest" value, the "minimum minimorum." However, by applying exactly the reasoning applied above, the truth of the following theorem is seen, which comprises I*c* and II:

III. *Assume $f(z)$ regular in a region S and on its boundary. Take the z-plane as the horizontal plane of a system of rectangular space coördinates, and erect in each point z of S and its boundary a perpendicular of length $|f(z)|$. The (continuous) surface generated by the end points of these perpendiculars has no maximum in the interior of S, and where it has a minimum in the interior, it reaches down to the z-plane.*

All of these theorems hold also for branches of multiple-valued analytic functions under proper restrictions for the region S; it is only necessary to stipulate that S shall be a closed region on the Riemann surface belonging to the multiple-valued function, and shall not contain a branch point of the Riemann surface in its interior or on its boundary. This follows from the fact that in such a region the corresponding branch of the multiple-valued function is nothing but an ordinary single-valued analytic function, to which therefore I*c* is applicable.

When one remembers: (1) that $df(z)/dz = f'(z)$ is an analytic function of z, regular everywhere where the single-valued analytic function $f(z)$ is regular, and that therefore I*c* may be applied to $f'(z)$; (2) that $|f'(z)|$ represents the factor of magnification at every point of the conformal mapping of a region in the z-plane upon the corresponding region of the $f(z)$-plane, we obtain the theorems IV and V:

IV. *For a region S, in every interior and boundary point of which the single-valued analytic function $f(z)$ is regular, the factor of magnification of the conformal mapping of the region S upon the corresponding region of the $f(z)$-plane has its largest value on the boundary; or it is constant for all points of the region S, and the function is of the form $f(z) = az + b$.*

The very last statement follows from the fact that $f'(z)$ is constant in this case. The coefficient a may be assumed different from zero, since $f(z) \equiv b$ does not yield any conformal representation. From the standpoint of conformal mapping, $f(z) = z$ is the simplest function.

V. *For a region S, in every interior and boundary point of which the single-valued analytic function $f(z)$ is regular and has its derivative different from zero, the factor of magnification has its smallest value on the boundary. It is constant only for $f(z) = az + b$.*

Since the values of z, for which $f'(z) = 0$, are points for which the conformal representation breaks down, we may combine IV and V by saying:

For any region S in which the conformal representation is not disturbed for any interior or boundary point, the factor of magnification has its largest and also its smallest value for points on the boundary of S; unless the function be of the type $f(z) = az + b$, in which case the factor of magnification is constant.

Any picture of a conformal mapping will serve to illustrate these theorems.

VI. In proving Ia or Ib, from which the other theorems are all derived, use is always made, as far as I know, of the analytic character of the function $f(z)$. That the analyticity is, in a certain sense, not essential, is easily established as follows:

Assume $f(z)$ to be a function of the complex variable $z = x + iy$ possessing the properties:

(1) There exists in the z-plane a two-dimensional region S for which $f(z)$ is single-valued and continuous;

(2) Every circle s lying entirely in S is transformed by $f(z)$ into a region s' of the $f(z)$-plane, so that interior points correspond to interior points, and boundary points to boundary points. (In case overlapping regions s' in the $f(z)$-plane are admitted, a slight modification in the statement of the correspondence of points is required, to avoid ambiguity.)

It is possible in various ways to impose these conditions on the function without making it analytic. They are satisfied, for example, in the whole finite complex plane, by $f(x + iy) = \alpha x + i\beta y$, where $\alpha \neq \beta$ are two real numbers, different from zero; this function represents stretching along the x-axis in a certain ratio and stretching along the y-axis in a different ratio. The function is not analytic.

After having chosen a region s in the z-plane, assume the corresponding s' mapped in the $f(z)$-plane, and describe in the $f(z)$-plane the circle of largest radius about the origin having points in common with s'; mark this point, which we denote by α (or one of these points, in case there are more than one), and let a be the corresponding point in s. Then a is a point on the boundary of s, while $|\alpha| = M$, the distance of α from the origin in the $f(z)$-plane, is the largest value attained by $|f(z)|$ for any point z lying in the interior or on the boundary of s. Therefore, *at least for a region in the z-plane satisfying* (1) *and* (2), Ia and Ib hold whether $f(z)$ is analytic or not.

To derive the corresponding theorem II on the smallest value of $|f(z)|$, two cases must be considered, according to whether s' contains or does not contain the origin of the $f(z)$-plane as an interior point.

The last discussion might equally well have been based on known theorems of two real functions of two real variables, making use, for example, of a theorem given in Osgood's "*Lehrbuch der Funktionentheorie,*" 1912, p. 71.

The theorems on the real and imaginary parts u and v of $f(x + iy) = u + iv$ may be just as easily derived under the assumptions (1) and (2).

§ II. Concerning the Smallest Integer $m!$ Divisible by a Given Integer n.

In connection with an investigation in number theory the solution of the following problem was needed:

I. For a given integer n to find the smallest integer $\mu(n) = m$ such that $m!$ is divisible by n. A direct method of determining m is required, by which all trials are eliminated.

This problem is closely related to the problem which is treated in many textbooks of elementary number theory:

Ia. For a given integer m and a given prime p to find the highest power of p contained as a factor in $m!$.[1]

The simple considerations leading to the solutions of these problems are of the same character, in the two cases, but the formulae which give the solution of the second problem do not yield a method of determining without trials the m of the first problem for a given n, even when n is restricted to be of the form p^α, p a prime.

Solution of I. All letters used denote positive integers or zero. For several forms of n the solution is obvious.

1. $\mu(1) = 1$; $\mu(n!) = n$.
2. $\mu(p) = p$, *p a prime.*
3. $\mu(p_1 \cdot p_2 \cdots p_\rho) = p_\rho$ *when* $p_1 < p_2 < \cdots < p_\rho$ *are distinct primes.*
4. $\mu(p^\alpha) = p\alpha$, *for p prime and* $\alpha \leqq p$.

The solution is not trivial for

5. $n = p^\alpha$, $\alpha \gtreqless p$ (thus including 2 and 4 as special cases).

In this case there are complications for the following reason. Let

$$m! = 1 \cdot 2 \cdots p \cdots (2p) \cdots (p \cdot p) \cdots ((p+1)p) \cdots (2p \cdot p) \cdots (p^2 \cdot p) \cdots m;$$

it is clear that every pth number contains p as a factor; but at the same time every (p^2)th number contains p a second time as a factor, every (p^3)th number contains p^3 as a factor, and so on. Denoting by $[a/b]$ the greatest integer k such that $bk \leqq a$, we see that $m!$ contains exactly p raised to the power

$$\left[\frac{m}{p}\right] + \left[\frac{m}{p^2}\right] + \left[\frac{m}{p^3}\right] + \cdots$$

as a factor, where the $[m/p^\nu]$ are integers decreasing with ν and of which only a certain number are different from zero.

This is the method by which Ia is usually proved. Taking $m = p^\alpha$, α any integer, we see that $(p^\alpha)!$ contains exactly the power $p^{1+p+p^2+\cdots+p^{\alpha-1}}$.[2] Therefore

$$(a) \qquad \mu(p^{1+p+p^2+\cdots+p^{\alpha-1}}) = p^\alpha.$$

The truth of the following statement will be obvious after writing it out for a special case, say $p = 5$, $\alpha = 4$, $\beta = 2$:

(b) Let g be any positive integer or zero, and $\alpha > \beta > 0$, p any prime; then in the product

[1] Compare, for example, L. W. Reid, *The Elements of the Theory of Algebraic Numbers*, 1910, p. 26, or R. D. Carmichael, *The Theory of Numbers*, 1914, p. 24.

[2] The notation $(p^\alpha)! \equiv 0 \bmod (p^{1+p+\cdots+p p^{\alpha-1}})$ has purposely been avoided throughout.

$$(g\cdot p^{\alpha}+1)\cdot(g\cdot p^{\alpha}+2)\cdots(g\cdot p^{\alpha}+p^{\beta})\cdot(g\cdot p^{\alpha}+p^{\beta}+1)$$
$$\cdots(g\cdot p^{\alpha}+2\cdot p^{\beta})\cdot(g\cdot p^{\alpha}+2\cdot p^{\beta}+1)$$
$$\cdots(g\cdot p^{\alpha}+3p^{\beta})\cdot\cdots\cdot(g\cdot p^{\alpha}+\overline{p-2}\cdot p^{\beta}+1)$$
$$\cdots(g\cdot p^{\alpha}+\overline{p-1}\cdot p^{\beta})\cdot(g\cdot p^{\alpha}+\overline{p-1}\cdot p^{\beta}+1)\cdots(g\cdot p^{\alpha}+p^{\beta+1})$$

each one of the $p-1$ products of p^{β} numbers each

$$(g\cdot p^{\alpha}+1)\cdots(g\cdot p^{\alpha}+p^{\beta});\ \cdots;\ (g\cdot p^{\alpha}+\overline{p-2}\cdot p^{\beta}+1)\cdots(g\cdot p^{\alpha}+\overline{p-1}\cdot p^{\beta})$$

contains as a factor exactly the same power of p as does the first group, that is $p^{1+p+\dots+p^{\beta-1}}$, while the last group of p factors, $(g\cdot p^{\alpha}+\overline{p-1}\cdot p^{\beta}+1)\cdots(g\cdot p^{\alpha}+p^{\beta+1})$, contains the factor $p\cdot p^{1+p+\dots+p^{\beta-1}}$, on account of the $p^{\beta+1}$ of the last factor.

Any positive integer may be written in the form $c_1\cdot p^{\delta}+c_2\cdot p^{\delta-1}+\cdots c_{\delta}\cdot p+c_{\delta+1}$, where for each $c: 0\leqq c<p$. It is clear that if m is to be the smallest integer for which $m!$ is divisible by a power of p, m will be divisible by p. Therefore m may be written

$$m=\gamma_1\cdot p^{\alpha_1}+\gamma_2\cdot p^{\alpha_2}+\cdots+\gamma_{\nu}\cdot p^{\alpha_{\nu}},$$

where

$$\alpha_1>\alpha_2>\cdots>\alpha_{\nu}>0$$

and

$$0<\gamma_i<p.$$

Then

$$m!=1\cdot 2\cdots(\gamma_1\cdot p^{\alpha_1})\cdot(\gamma_1\cdot p^{\alpha_1}+1)\cdots(\gamma_1\cdot p^{\alpha_1}+\gamma_2\cdot p^{\alpha_2})\cdot(\gamma_1\cdot p^{\alpha_1}+\gamma_2\cdot p^{\alpha_2}+1)$$
$$\cdots(\gamma_1\cdot p^{\alpha_1}+\gamma_2\cdot p^{\alpha_2}+\gamma_3\cdot p^{\alpha_3})\cdots(\gamma_1\cdot p^{\alpha_1}+\gamma_2\cdot p^{\alpha_2}+\cdots+\gamma_{\nu-1}\cdot p^{\alpha_{\nu-1}}+1)$$
$$\cdots(\gamma_1\cdot p^{\alpha_1}+\gamma_2\cdot p^{\alpha_2}+\cdots+\gamma_{\nu-1}\cdot p^{\alpha_{\nu-1}}+\gamma_{\nu}\cdot p^{\alpha_{\nu}}).$$

In $1\cdot 2\cdots(\gamma_1\cdot p^{\alpha_1})$ exactly $p^{\gamma_1(1+p+p^2+\dots+p^{\alpha_1-1})}$ is contained as a factor; in $(\gamma_1\cdot p^{\alpha_1}+1)\cdots(\gamma_1\cdot p^{\alpha_1}+\gamma_2\cdot p^{\alpha_2})$ exactly $p^{\gamma_2(1+p+p^2+\dots+p^{\alpha_2-1})}$ is contained, and so on; finally, in

$$(\gamma_1\cdot p^{\alpha_1}+\cdots+\gamma_{\nu-1}\cdot p^{\alpha_{\nu-1}}+1)\cdots(\gamma_1\cdot p^{\alpha_1}+\cdots+\gamma_{\nu-1}\cdot p^{\alpha_{\nu-1}}+\gamma_{\nu}\cdot p^{\alpha_{\nu}})$$

we have exactly the factor $p^{\gamma_{\nu}(1+p+p^2+\dots+p^{\alpha_{\nu}-1})}$.

Therefore $m!$ contains exactly p raised to the power

$$\sum_{\rho=1}^{\nu}\gamma_{\rho}(1+p+p^2+\cdots+p^{\alpha_{\rho}-1}).$$

We have thus obtained (with a slight change in notation) the following formula which solves 5., *provided we can always write the given exponent α in the form*

$$\alpha = \sum_{\rho=1}^{\nu} \gamma_\rho (1 + p + p^2 + \cdots + p^{\beta_\rho}),$$

with all γ *satisfying* $0 < \gamma < p$ *and* $\beta_1 > \beta_2 > \cdots > \beta_\rho \geqq 0$:

$$(c) \qquad \mu\left(p^{\sum_{\rho=1}^{\nu} \gamma_\rho (1+p+p^2+\dots+p^{\beta_\rho})}\right) = \sum_{\rho=1}^{\nu} \gamma_\rho \cdot p^{\beta_\rho + 1}.$$

We must next examine whether an integer α may always be written in the form $\alpha = \gamma_1(1 + p + \cdots + p^{\beta_1}) + \cdots + \gamma_\nu(1 + p + \cdots + p^{\beta_\nu})$, where $\beta_1 > \beta_2 > \cdots > \beta_\nu \geqq 0$ and where each γ satisfies: $0 < \gamma < p$. It will be seen that there are exceptional cases to be taken care of.[1]

Let

$$a_1 = 1, \quad a_2 = 1 + p, \quad \cdots, \quad a_{\rho+1} = 1 + p + p^2 + \cdots + p^\rho, \quad \cdots,$$

then we have the recurrence relation

$$a_{\rho+1} = p \cdot a_\rho + 1, \quad a_1 = 1.$$

Assume $a_\nu \leqq \alpha < a_{\nu+1}$, then

$$\alpha = k_\nu a_\nu + r_\nu,$$

where

$$0 < k_\nu < p,$$

except for the possible combination $k_\nu = p$, $r_\nu = 0$. Assume for the moment $r_\nu > 0$, then there is an a_σ such that $a_\sigma \leqq r_\nu < a_{\sigma+1}$, and

$$r_\nu = k_{\nu-1} a_{\nu-1} + r_{\nu-1},$$

where

$$0 < k_{\nu-1} < p,$$

except for the combination $k_{\nu-1} = p$, $r_{\nu-1} = 0$. Continuing in this fashion, we see that for every integer α

$$\alpha = \gamma_1(1 + p + \cdots + p^{\beta_1}) + \cdots + \gamma_\nu(1 + p + \cdots + p^{\beta_\nu}),$$

where $\beta_1 > \beta_2 > \cdots > \beta_\nu \geqq 0$, but $0 < \gamma < p$ only with the restriction that the last γ may be equal to p. This exception arises from the fact that while for every $m!$ a certain p^α exists which is the highest power of p contained as a factor in $m!$, for a given p^α there may not exist an m such that $m!$ is divisible by p^α and by no higher power of p (for example $p = 2$, $\alpha = 6$).

To show that our method of determining $\mu(p^\alpha)$ by means of (c) holds also in case the last coefficient γ is equal to p, we proceed as follows:

First assume

$$\alpha = \gamma_1(1 + p + \cdots + p^{\beta_1}) + p(1 + p + \cdots + p^{\beta_1 - 1})$$

[1] These exceptional cases were overlooked by J. Neuberg, *Mathesis*, 1887, Vol. VII, p. 68: But for them, our problem would be solved in a very few lines from the known methods of solving problem I*a*.

or

$$\alpha + 1 = (\gamma_1 + 1)(1 + p + \cdots + p^{\beta_1}),$$

assuming for the moment $\gamma_1 + 1 < p$. This increases α by one unit, but we shall show that

$$\mu(p^{\alpha+1}) = \mu(p^\alpha).$$

Clearly $\mu(p^{\alpha+1}) = (\gamma_1 + 1)\cdot p^{\beta_1+1}$, and $[(\gamma_1 + 1)\cdot p^{\beta_1+1}]!$ contains exactly the factor $p^{\alpha+1}$. Now let m_1 be the smallest number such that $m_1!$ has the factor p^α; then $m_1 = \mu(p^{\alpha+1})$ because if we omit the last factor of $[\mu(p^{\alpha+1})]!$, that is $(\gamma_1 + 1)\cdot p^{\beta_1+1}$, the exponent of the highest power of p contained as a factor in the product is reduced by $\beta_1 + 1 \geqq 2$ units at least. It is easy to see that if α contains terms preceding $\gamma_1(1 + p + \cdots + p^{\beta_1})$ the argument holds unchanged, on account of the statement (b).

We must still free ourselves from the assumption $\gamma_1 + 1 < p$. But when $\gamma_1 + 1 = p$, our new last coefficient is equal to p and we may repeat the process just indicated; the original α is now increased to $\alpha + 2$, but still $\mu(p^{\alpha+2}) = \mu(p^\alpha)$.

An example will suffice to illustrate conditions: Take $p = 3$, $\alpha = 37$. We have $37 = 2\cdot(1 + 3 + 9) + 2\cdot(1 + 3) + 3\cdot 1$; replace this by $2\cdot(1 + 3 + 9) + 3\cdot(1 + 3) = 38$; and again by $3\cdot(1 + 3 + 9) = 39$; and finally by $1\cdot(1 + 3 + 9 + 27) = 40$. Then $(1\cdot 3^4)! = (81)!$ is the smallest number $m!$ containing 3^{37} as a factor.

It contains also 3^{40} as a factor, since $\mu(3^{37}) = \mu(3^{38}) = \mu(3^{39}) = \mu(3^{40}) = 81$, as can be immediately verified.

However, since the relation

$$\alpha = \gamma_1(1 + p + \cdots + p^{\beta_1}) + p\cdot(1 + p + \cdots + p^{\beta_1 - 1}),$$

when treated as if the rule given by (c) still held although $\gamma_2 = p$, will give us $\gamma_1\cdot p^{\beta_1+1} + p\cdot p^{\beta_1} = (\gamma_1 + 1)\cdot p^{\beta_1+1}$, that is, the same value which we get by applying (c) to $\alpha + 1 = (\gamma_1 + 1)\cdot(1 + p + \cdots + p^{\beta_1})$, we shall have the theorem:

Theorem: *Assume α written in the form*

$$\alpha = \gamma_1(1 + p + \cdots + p^{\beta_1}) + \gamma_2(1 + p + \cdots + p^{\beta_2}) + \cdots + \gamma_\nu(1 + p + \cdots + p^{\beta_\nu}),$$

where $\beta_1 > \beta_2 > \cdots > \beta_\nu \geqq 0$, $0 < \gamma_\rho < p$ *for* $\rho = 1, 2, \cdots, \nu - 1$, *but* $0 < \gamma_\nu \leqq p$. *Then*

$$\mu(p^\alpha) = \gamma_1\cdot p^{\beta_1+1} + \gamma_2\cdot p^{\beta_2+1} + \cdots + \gamma_\nu\cdot p^{\beta_\nu+1}$$
$$= (p - 1)\cdot\alpha + (\gamma_1 + \gamma_2 + \cdots + \gamma_\nu).$$

The very last equality is verified immediately by substituting for α the expression given in the theorem.

Example: $p^{\alpha} = 5^{1092}$.

$$\alpha = 1\cdot(1 + 5 + \cdots + 5^4) + 1\cdot(1 + 5 + 5^2 + 5^3) + 5\cdot(1 + 5 + 5^2).$$

$$\mu(5^{1092}) = 1.5^5 + 1\cdot 5^4 + 5\cdot 5^3 = 4.1092 + (1 + 1 + 5) = 4375.$$

It may be left to the reader to show that the last theorem is equivalent to the following

RULE: *To find* $\mu(p^{\alpha})$, *form a sequence of numbers by means of the recurrence formula*

$$a_{\rho+1} = p\cdot a_{\rho} + 1, \quad a_1 = 1.$$

Assume $a_{\nu} \leqq \alpha < a_{\nu+1}$ *and carry out the following divisions until we meet a remainder zero:*

$$\begin{aligned} \alpha &= k_{\nu} \cdot a_{\nu} + r_{\nu}, & 0 < r_{\nu} < a_{\nu}, \\ r_{\nu} &= k_{\nu-1}\cdot a_{\nu-1} + r_{\nu-1}, & 0 < r_{\nu-1} < a_{\nu-1}, \\ &\cdots\cdots\cdots\cdots \\ r_{\lambda+2} &= k_{\lambda+1}\cdot a_{\lambda+1} + r_{\lambda+1}, & 0 < r_{\lambda+1} < a_{\lambda+1}, \\ r_{\lambda+1} &= k_{\lambda} \cdot a_{\lambda}. \end{aligned}$$

Then

$$\mu(p^{\alpha}) = (p - 1)\cdot\alpha + (k_{\nu} + k_{\nu-1} + \cdots + k_{\lambda+1} + k_{\lambda}).$$

This should be compared with the following rule for solving problem I*a* (see Carmichael, *loc. cit.*, p. 26):

To find the highest power p^{α} of a prime p contained in $m!$, write m in the form $m = q_1\cdot p^{\alpha_1} + q_2\cdot p^{\alpha_2} + \cdots + q_{\tau}\cdot p^{\alpha_{\tau}}$, $\alpha_1 > \alpha_2 > \cdots > \alpha_{\tau} \geqq 0$, all q satisfying $0 < q < p$; then

$$\alpha = \frac{m - (q_1 + q_2 + \cdots + q_{\tau})}{p - 1}.$$

Example 1: $p^{\alpha} = 3^{1000}$.

$$a_1 = 1;\ a_2 = 3\cdot a_1 + 1 = 4;\ a_3 = 3\cdot a_2 + 1 = 13;\ a_4 = 3\cdot a_3 + 1 = 40;$$

$$a_5 = 3\cdot a_4 + 1 = 121;\ a_6 = 3\cdot a_5 + 1 = 364;\ a_7 = 3\cdot a_6 + 1 = 1093,$$

and

$$\alpha = 2\cdot a_6 + 272,\ 272 = 2\cdot a_5 + 30,\ 30 = 2\cdot a_3 + 4,\ 4 = 1\cdot a_2.$$

Hence

$$\mu(3^{1000}) = 2\cdot 1000 + (2 + 2 + 2 + 1) = 2007.$$

Example 2: $p^{\alpha} = 5^{100000}$.

$$a_1 = 1; a_2 = 6; a_3 = 31; a_4 = 156; a_5 = 781; a_6 = 3906; a_7 = 19531; a_8 = 97651.$$

Then

$$100000 = 1\cdot 97651 + 2349,\ 2349 = 3\cdot 781 + 6,\ 6 = 1\cdot 6;$$

$$\mu(5^{100000}) = 4\cdot 100000 + (1 + 3 + 1) = 400005.$$

6. $n = p_1^{\epsilon_1}\cdot p_2^{\epsilon_2}\cdots p_{\nu}^{\epsilon_{\nu}}$, *where the p's are distinct primes.* (General case.)

This case reduces immediately to 5., because if $m!$ contains the factors $p_1^{\epsilon_1}, \cdots, p_\nu^{\epsilon_\nu}$, it must also have their product as a factor.

Denoting by max $[\mu(p_1^{\epsilon_1}), \mu(p_2^{\epsilon_\nu}), \cdots, \mu(p_\nu^{\epsilon_\nu})]$ the largest of the numbers $\mu(p_1^{\epsilon_1}), \cdots, \mu(p_\nu^{\epsilon_\nu})$, we have the theorem (which of course includes 3. as a special case):

THEOREM: $\mu(p_1^{\epsilon_1}\cdot p_2^{\epsilon_2}\cdots p_\nu^{\epsilon_\nu}) = \max [\mu(p_1^{\epsilon_1}), \mu(p_2^{\epsilon_2}), \cdots, \mu(p_\nu^{\epsilon_\nu})]$.

Example: $n = 3^2\cdot 5^{29}\cdot 11^{19}\cdot 113$.

$\mu(3^2) = 2\cdot 3 = 6$; $\mu(5^{29}) = 4\cdot 5^2 + 5\cdot 5 = 125$, because $29 = 4(1+5) + 5.1$; $\mu(11^{19}) = 1\cdot 11^2 + 7\cdot 11 = 198$, because $19 = 1\cdot(1+11) + 7\cdot 1$; $\mu(113) = 113$, because 113 is a prime number; therefore $\mu(3^2\cdot 5^{29}\cdot 11^{19}\cdot 113) = 198$.

Considering $\mu(n)$ as a function of n, one sees that it has, like most number theoretic functions, a very irregular behavior:

$$n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24,$$

$$\mu(n) = 1, 2, 3, 4, 5, 3, 7, 4, 6, \ 5, 11, \ 4, 13, \ 7, \ 5, \ 6, 17, \ 6, 19, \ 5, \ 7, 11, 23, \ 4.$$

BOOK REVIEWS AND NOTICES.

SEND ALL COMMUNICATIONS ABOUT BOOKS TO W. H. BUSSEY, University of Minnesota.

Descriptive Geometry. By ERVIN KENISON and HARRY CYRUS BRADLEY. The Macmillan Company, New York, 1917. x + 287 pages. $2.00.

The book under review confines itself to a treatment of that branch of descriptive geometry which is known as the method of double orthographic projection or more simply as the Mongean method.

The very brief introduction which precedes Chapter I is scarcely sufficient to give the reader an adequate notion of the purposes and nature of descriptive geometry. Exception might be taken to the statement: "Its operations are not strictly mathematical." For, aside from the more or less precise mechanical operation involved in the use of the drawing instruments, the methods which descriptive geometry continually applies are, according to Loria, only such as are taught by the old Euclidean geometry and the modern projective geometry, and its processes are so very exact that it is comparable with algebra and analysis.[1]

The first seventeen chapters are concerned principally with the method of representing points, lines, and planes of space, and of solving, by means of plane constructions, the problems of space which involve points, lines, and planes. The great number of problems which have been solved illustrate the more important processes of the Mongean method and give the student a very good notion of the operations of this method. The division of problems into chapters might suggest an attempt at classification. However, no explicit statements have been made which would lead the student to suspect that all the problems

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[1] See Loria, *Vorlesungen über Darstellende Geometrie*, preface to Vol. I.